Int J Theor Phys (2009) 48: 2757–2776
DOI 10.1007/s10773-009-0066-2

An Alternative Model of the Damped Harmonic Oscillator Under the Influence of External Force

M. Sebawe Abdalla · Nour Al-Ismael

Received: 23 March 2009 / Accepted: 9 June 2009 / Published online: 18 June 2009

Abstract In this paper we introduce the modified time-dependent damped harmonic oscillator. An exact solution of the wave function for both Schrödinger picture and coherent state representation are given. The linear and quadratic invariants are also discussed and the corresponding eigenvalues and eigenfunctions are calculated. The Hamiltonian is transformed to $SU(1,1)$ Lie algebra and an application to the generalized coherent state is discussed. It has been shown that when the system is under critical damping case the *maximum* squeezing is observed in the first quadrature F_x. However, for the overcritical damping case the *maximum* squeezing occurs in the second quadrature F_y. Also it has been shown that the system for both cases is sensitive to the variation in the coherent state phase.

Keywords Wave function · Constants of the motion · Squeezing phenomenon

1 Introduction

In the present communication we reconsider the problem of time-dependent harmonic oscillator which has been extensively studied in the mid of the last century, see for example [1–12]. The question which may arise why one comes back to study one of the old problem and try to resurrect it. The answer is not just a matter to reconsider certain problem and try to recover some of the gaps in it. In fact this particular problem (here we refer to the time-dependent harmonic oscillator) has opened the door to consider different aspects in the classical as well as in the quantum mechanics. For instance, the existence of the time-dependent mass in the harmonic oscillator leads to the appearance of the second harmonic generation, and consequently the system turned to degenerate parametric amplifier model [13–15]. Also, the realization of the group symmetry in this system gives us the opportunity to consider the Lie algebraic treatment of such problem [16–19]. Furthermore, the observation of the nonclassical phenomena in the laboratory and particularly the squeezing

M. Sebawe Abdalla (✉) · N. Al-Ismael
Mathematics Department, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: m.sebaweh@physics.org

phenomenon and its connection to the second harmonic generation encouraged us to go back to the problem. There is no doubt the appearance of the second harmonic generation would lead us to think of the nonclassical properties for such system. Therefore, as apart of our duty in this context is to discuss the squeezing phenomenon, however, from point of view of the Lie Algebra, $SU(1,1)$ [20–22]. Here we may refer to the previous works for the problem of quantizing the damped motion of a particle in a quadratic field. The problem is usually deal with an oscillator with constant mass and stiffness placed in the presence of a dissipative force $F=-\gamma\dot{X}$, where γ being constant. In addition, there exists a substantial body of work concerning the study of a classical, undamped harmonic oscillator with arbitrary dependence in its parameters, see for example [1–3, 23, 24]. In the present work we aim at unifying these views in order to provide a treatment of a quantal oscillator in the presence of a dissipation mechanism, in the most general situation in which the mass is time-dependent. Our consideration will be extended to include the time-dependent driving force which acts on the Hamiltonian model, see for example [11, 12, 25, 26]. In what follows we concentrate on a particular time-dependent mass law which is given by

$$M(t)=m\exp\left[-2\sqrt{\omega^2t^2+\delta^2}\right], \tag{1.1}$$

where the time-dependent mass reflects the modified damped harmonic oscillator [24]. In the above equation we assume that the mass is decaying with a factor equal to the system frequency ω (say). This in fact pushed us to add the shifted parameter δ that to avoid the critical decaying case. The Hamiltonian for a linear oscillator with time-dependent mass spring constant is given by

$$\hat{H}(t)=\frac{\hat{P}^2}{2M(t)}+\frac{1}{2}\omega^2M(t)\hat{Q}^2, \tag{1.2}$$

where $\hat{P}$ and $\hat{Q}$ are the dynamical operators which represent the dimension of momentum and coordinates, respectively and satisfy the commutation relation $[\hat{Q},\hat{P}]=i\hbar$. ω is the frequency of the system and $M(t)$ is a time-dependent mass. In presence of an external driving force, the Hamiltonian (1.2) can be written thus

$$\hat{H}(t)=\frac{\hat{P}^2}{2M(t)}+\frac{1}{2}\omega^2M(t)\hat{Q}^2+E(t)\hat{Q}, \tag{1.3}$$

where $E(t)$ is any time-dependent function which usually represents the electric field [27, 28]. In our study we handle the problem from quantum mechanics point of view that to fill the gap of the previous work, more precisely the nonclassical effect. In the meantime we extend our interest to include the wave function in both non-stationary number and coherent states. Moreover, we seek the constants of the motion where we introduce the linear and the quadratic invariants. Also, as a relationship between the time-dependent system and the Lie algebra, we introduce the solution for the equations of motion in terms of $SU(1,1)$ Lie algebra generators. This gives us an advantage to discuss the phenomenon of squeezing in terms of the Perelomov $SU(1,1)$ coherent state. Therefore the paper is organized as follows: In Sect. 2 we give the explicit expression of the wave function in Schrödinger picture as well as in the coherent state representation. Section 3 is devoted to introduce the accurate definition of the creation and annihilation operators from which the Hamiltonian (1.3) can be diagonalized. In Sect. 4 we introduce the constants of motion (linear and quadratic invariants) and obtain their eigenvalues and the corresponding eigenfunctions. Finally in Sect. 5 we give an application for the present system by discussing the phenomenon of squeezing from the Perelomov coherent state point of view. Our conclusion is given in Sect. 6.

2 The Wave Function

As we have stated before one of our task is to calculate the wave function in the Schrödinger picture and to calculate the corresponding coherent state. For this reason we devote the present section to obtain the exact expression of the wave function in the number state and in the coherent state.

2.1 Schrödinger Picture

To find the wave function in the number state representation we make our starting point the canonical transformation, $\hat{Q}=\sqrt{m/M(t)}\hat{q}$ and $\hat{P}=\sqrt{M(t)/m}\hat{p}$, with the properties $[\hat{Q},\hat{P}]=[\hat{q},\hat{p}]=i\hbar$. This means that we have to rewrite the Hamiltonian (1.3) in the form

$$\hat{H}(t)=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{q}^2+\frac{\Gamma(t)}{2}\left(\hat{p}\hat{q}+\hat{q}\hat{p}\right)+\bar{E}(t)\hat{q}, \tag{2.1}$$

where $\Gamma(t)=d(\ln\sqrt{M(t)})/dt$ and $\bar{E}(t)=E(t)\exp[-\sqrt{\omega^2t^2+\delta^2}]$. The time-dependent Schrödinger equation is given by

$$\hat{H}(t)\psi(q,t)=i\hbar\frac{\partial}{\partial t}\psi(q,t). \tag{2.2}$$

Now if we substitute (2.1) into (2.2) then the wave function takes the form

$$\frac{\partial^2\psi}{\partial q^2}-\frac{m^2\omega^2}{\hbar^2}q^2\psi+\frac{im\Gamma(t)}{\hbar}\left(2q\frac{\partial\psi}{\partial q}+\psi\right)-\frac{2m}{\hbar^2}\bar{E}(t)q\psi=-\frac{2im}{\hbar}\frac{\partial\psi}{\partial t}. \tag{2.3}$$

Furthermore, if one makes the substitution

$$\hat{q}=x+\zeta(t),\qquad \psi(q,t)\equiv\eta(x,t),\qquad d\psi=d\eta, \tag{2.4}$$

where the function $\zeta(t)$ is suitably chosen, then

$$\frac{\partial\eta}{\partial x}=\frac{\partial\psi}{\partial q},\qquad \frac{\partial\psi}{\partial t}=\frac{\partial\eta}{\partial t}-\dot{\zeta}\frac{\partial\eta}{\partial x},\qquad \frac{\partial^2\psi}{\partial q^2}=\frac{\partial^2\eta}{\partial x^2} \tag{2.5}$$

and consequently (2.3) reduces to the form

$$\frac{\partial^2\eta}{\partial x^2}-\frac{m}{\hbar^2}\left[2\bar{E}\,(x+\zeta)+m\omega^2\,(x+\zeta)^2\right]\eta+\frac{im\Gamma(t)}{\hbar}\left(2\,(x+\zeta)\,\frac{\partial}{\partial x}+1\right)\eta-\frac{2im}{\hbar}\dot{\zeta}\frac{\partial\eta}{\partial x}$$
$$=-\frac{2im}{\hbar}\frac{\partial\eta}{\partial t}. \tag{2.6}$$

Moreover, we use the transformation

$$\eta(x,t)=\Phi(y,t),\qquad y=u(t)x\quad\text{and}\quad u(t)=\frac{1}{|R(t)|}, \tag{2.7}$$

where we define $R(t)=(\lambda\cosh\phi(t)-i\sinh\phi(t))$, $\phi(t)=\frac{1}{2}\tanh^{-1}(\frac{\Gamma(t)}{\omega})$ and $\lambda=\sqrt{(\delta-\frac{1}{2})/(\delta+\frac{1}{2})}$.

In this case (2.6) can be written as follows

$$\frac{\partial^2\Phi}{\partial y^2} - \frac{m}{\hbar^2 u^4}\left(2u\bar{E}\,(y+u\zeta) + m\omega^2\,(y+u\zeta)^2\right)\Phi + \frac{2im}{\hbar u}\left(\frac{\Gamma(t)}{u}\,(y+u\zeta) - \dot{\zeta} + \frac{\dot{u}}{u^2}y\right)\frac{\partial\Phi}{\partial y} = -\frac{im}{\hbar u^2}\left(2\frac{\partial\Phi}{\partial t} + \Gamma(t)\Phi\right). \tag{2.8}$$

Now we seek a separation of the form

$$\Phi = Y(y)T(t)\exp\left[-\frac{im}{2\hbar}\left(V(t)y^2 + 2W(t)y\right)\right], \tag{2.9}$$

where $V(t)$ and $W(t)$ are time-dependent functions given by

$$V(t) = \frac{\Gamma(t)u + \dot{u}}{u^3}, \quad \text{and} \quad W(t) = \frac{\Gamma(t)\zeta - \dot{\zeta}}{u(t)}. \tag{2.10}$$

Substituting (2.9) and (2.10) into (2.8) and after some calculations we can simplify the result to take the form

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{m^2\omega^2}{\hbar^2}\left(\frac{\dot{V}(t)}{\omega^2u^2} + \frac{V^2(t)}{\omega^2} - \frac{1}{u^4}\right)y^2 - \frac{2m}{\hbar^2 u}\left(\frac{\bar{E} + m\omega^2\zeta}{u^2} - \frac{m\dot{W}}{u} - muV(t)W(t)\right)y = -\frac{2im}{\hbar u^2}\left(\frac{1}{T}\frac{dT}{dt} + \frac{\Gamma(t) - V(t)u^2}{2}\right) + \frac{m^2}{\hbar^2 u^2}\left(2\frac{\bar{E}}{m}\zeta + \omega^2\zeta^2 - u^2W^2(t)\right). \tag{2.11}$$

To complete the separation of the variable we choose $\zeta(t)$ to make the coefficient of y in (2.11) vanishes. Simple calculations lead us to have the driven linear equation for $\zeta(t)$, thus

$$\ddot{\zeta} + \left(\omega^2 - \dot{\Gamma}(t) - \Gamma^2(t)\right)\zeta = -\frac{\bar{E}}{m}. \tag{2.12}$$

Using the fact that

$$\dot{V} + \left(V^2 + \lambda^2\omega^2\right)u^2 = \frac{\omega^2}{u^2}, \tag{2.13}$$

the wave function (2.11) separates to

$$\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{m^2\omega^2\lambda^2}{\hbar^2}y^2 = -\frac{2im}{\hbar}\,|R(t)|^2\left(\frac{1}{T}\frac{dT}{dt} - \frac{1}{2u}\frac{du}{dt}\right) + \frac{m^2}{\hbar^2}\,|R(t)|^2\left(2\frac{\bar{E}}{m}\zeta + \omega^2\zeta^2 - \left(\Gamma(t)\zeta - \dot{\zeta}\right)^2\right). \tag{2.14}$$

After a straightforward calculation we find, with K a constant of separation

$$\frac{d^2Y}{dy^2}+\left(K-\frac{m^2\omega^2\lambda^2}{\hbar^2}y^2\right)Y=0, \tag{2.15}$$

$$\frac{1}{T}\frac{dT}{dt}-\frac{1}{2u}\frac{du}{dt}+\left[\frac{i\hbar K}{2m\,|R(t)|^2}+\frac{im}{2\hbar}\left(\frac{2E}{m}\zeta+\omega^2\zeta^2-\frac{W^2(t)}{|R(t)|^2}\right)\right]=0. \tag{2.16}$$

Equation (2.15) is ordinary differential equation for the harmonic oscillator and requires the quantization

$$K=\frac{m\omega\lambda}{\hbar}(2n+1)\qquad n=0,1,2,\ldots \tag{2.17}$$

Therefore (2.9) may be written as

$$\begin{aligned}\Phi(y,t)=\frac{N}{\sqrt{|R(t)|}}H_n\left[\sqrt{\frac{m\omega\lambda}{\hbar}}y\right]\exp\left[-\frac{m\omega\lambda}{2\hbar}y^2\right]\\ \times\exp\left[-\frac{im}{2\hbar}\left(V(t)y^2+2W(t)y\right)\right]\\ \times\exp\left[-i\omega\lambda\left(n+\frac{1}{2}\right)\int_0^t|R(t)|^{-2}\,dt+I(t)\right],\end{aligned} \tag{2.18}$$

where N is the normalization constant and

$$I(t)=\int_0^t\left\{\zeta(t)\left[\frac{2\bar{E}}{m}+\omega^2\zeta(t)\right]-\frac{W^2(t)}{|R(t)|^2}\right\}dt. \tag{2.19}$$

Thus the wave function in its final form is

$$\begin{aligned}\psi_n(q,t)=\left(\frac{m\omega\lambda}{\pi\hbar|R(t)|^2}\right)^{\frac{1}{4}}(n!)^{-\frac{1}{2}}2^{-\frac{n}{2}}H_n\left(\sqrt{\frac{m\omega\lambda}{\hbar|R(t)|^2}}(q-\zeta(t))\right)\\ \times\exp\left[-\frac{m}{2\hbar}\left(\frac{(\omega\lambda+iV(t))}{|R(t)|^2}(q-\zeta(t))^2+\frac{2iW(t)}{|R(t)|}(q-\zeta(t))\right)\right]\\ \times\exp\left[-i\omega\left(n+\frac{1}{2}\right)\left(\beta\sinh^{-1}\left(\frac{\omega t}{\delta}\right)-\tan^{-1}\left(\frac{\tanh\phi}{\lambda}\right)\right)-\frac{im}{2\hbar}I(t)\right],\end{aligned} \tag{2.20}$$

where $\beta=\sqrt{\delta^2-\frac{1}{4}}$. It should be noted that the solution we have obtained is only valid within the interval $\delta\varepsilon(-\infty,-\frac{1}{2})\cup(\frac{1}{2},\infty)$ where the under critical damping case occurred.

2.2 The coherent state representation

To obtain the wave function in the coherent state representation we have to use the effect of the Glauber displacement operator on the vacuum state $\hat{D}(\alpha)|0\rangle=\exp(\hat{A}^\dagger\alpha-\hat{A}\alpha^*)|0\rangle$, viz

$$|\alpha\rangle=\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.21}$$

Therefore, if one uses (2.20) and (2.21) thus we obtain

$$\psi_\alpha(q,t) = \left(\frac{m\omega\lambda}{\pi\hbar|R(t)|^2}\right)^{\frac{1}{4}} \exp\left(\sqrt{\frac{2m\omega\lambda}{\hbar}}\frac{(q-\zeta(t))}{|R(t)|}\alpha(t)\right) \times \exp\left[-\frac{m}{2\hbar}\left(\frac{(\omega\lambda + iV(t))}{|R(t)|^2}(q-\zeta(t))^2 + \frac{2iW(t)}{|R(t)|}(q-\zeta(t))\right)\right] \times \exp\left[-\frac{1}{2}\left(\alpha^2(t) + |\alpha|^2 + \frac{im}{\hbar}I(t)\right)\right], \tag{2.22}$$

where we have dropped the zero point energy and defined

$$\alpha(t) = \alpha(0)\exp\left[-i\omega\left(\beta\sinh^{-1}\left(\frac{\omega t}{\delta}\right) - \tan^{-1}\left(\frac{\tanh\phi}{\lambda}\right)\right)\right]. \tag{2.23}$$

As a special case if we take the driving force $\bar{E} = 0$, then (2.22) reduces to

$$\psi_\alpha(q,t) = \left(\frac{m\omega\lambda}{\pi\hbar|R(t)|^2}\right)^{\frac{1}{4}} \exp\left[-\frac{m\omega}{2\hbar|R(t)|^2}\left(\lambda + \frac{i}{2}(\lambda^2+1)\sinh 2\phi\right)q^2\right] \times \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha^n(t)}{n!}2^{-n/2}H_n\left(\sqrt{\frac{m\omega\lambda}{\hbar|R(t)|^2}}q\right) \times \exp\left[in\tan^{-1}\left(\frac{\tanh\phi}{\lambda}\right)\right], \tag{2.24}$$

where the parameter $\alpha(t)$ in this case becomes

$$\alpha(t) = \alpha(0)\exp\left[-i\omega\left(\beta\sinh^{-1}\left(\frac{\omega t}{\delta}\right)\right)\right]. \tag{2.25}$$

In what follows we shall turn our attention to diagonalize the Hamiltonian model (2.1) using the result obtained in this section.

3 The Diagonalized Hamiltonian

In order to diagonalize the Hamiltonian model (2.1) we have to introduce a new definition of the boson operator. To achieve this goal one has to differentiate the wave function in the coherent state equation (2.22) or (2.24) in absence of the driving force. In presence of the driving force case and after rearrangement of the definition of the coherent state, the annihilation operator takes the form

$$\hat{A}(t)\,(2m\omega\lambda\hbar)^{\frac{1}{2}} = \left[m\omega\left(J^*(t)\hat{q}(t) + i\mathcal{K}(t)\right) + iR(t)\hat{p}(t)\right], \tag{3.1}$$

where $\hat{A}(t)$ with its adjoint satisfy the commutation relation $[\hat{A}(t), \hat{A}^\dagger(t)] = 1$. In (3.1) we have used the abbreviations

$$\mathcal{K}(t) = \frac{1}{m\omega}\int_0^t \bar{E}(\tau)R(\tau)\exp[i(\theta(\tau)-\theta(t))]\,d\tau,$$

$$J(t) = (\cosh\phi(t) - i\lambda\sinh\phi(t)), \quad \text{and} \quad \theta(t) = \beta\sinh^{-1}\left(\frac{\omega t}{\delta}\right). \tag{3.2}$$

We now substitute the operator $\hat{A}(t)$ with its complex conjugate into the Hamiltonian (2.1), thus we have

$$\begin{aligned}
\frac{\hat{H}(t)}{\hbar} &= \frac{\Omega}{2\lambda}\left(\lambda^2+1\right)\left(\hat{A}^\dagger(t)\hat{A}(t)+\frac{1}{2}\right)+\frac{\Omega}{4\lambda}\left(\lambda^2-1\right)\left(\hat{A}^2(t)+\hat{A}^{\dagger 2}(t)\right)\\
&+\frac{i\Omega}{2\lambda}\sqrt{\frac{m\omega}{2\lambda\hbar}}\Big[\left\{(1+\lambda^2)\mathcal{K}^*(t)+(1-\lambda^2)\mathcal{K}(t)\right\}\hat{A}(t)\\
&-\left\{(1-\lambda^2)\mathcal{K}^*(t)+(1+\lambda^2)\mathcal{K}(t)\right\}\hat{A}^\dagger(t)\Big]\\
&+\frac{m\omega\Omega}{8\hbar\lambda^2}\left[\left(\mathcal{K}(t)+\mathcal{K}^*(t)\right)^2+\lambda^2\left(\mathcal{K}(t)-\mathcal{K}^*(t)\right)^2\right]\\
&+\bar{E}(t)\Big[(2m\hbar\omega\lambda)^{-\frac{1}{2}}\left(\hat{A}(t)R^*+\hat{A}^\dagger(t)R\right)\\
&-\frac{i}{2\lambda\hbar}\left(\mathcal{K}(t)R^*-\mathcal{K}^*(t)R\right)\Big].
\end{aligned} \tag{3.3}$$

Since the operator (3.1) is explicitly time-dependent, therefore we have to add the first derivative of the generating function $F_2(\hat{A}(t), \hat{A}^\dagger(t), t)$ (say) into the Hamiltonian (3.3). In this case and after straightforward calculation the first derivative of the generating function takes the form

$$\begin{aligned}
\frac{\partial F_2}{\partial t} &= \dot{\phi}(t)\frac{(1-\lambda^2)}{2\lambda}\left(\hat{A}^\dagger(t)\hat{A}(t)+\frac{1}{2}\right)-\dot{\phi}(t)\frac{(1+\lambda^2)}{4\lambda}\left(\hat{A}^2(t)+\hat{A}^{\dagger 2}(t)\right)\\
&-\frac{i}{2\lambda}\sqrt{\frac{m\omega}{2\lambda\hbar}}\left[\dot{\phi}(t)\mathcal{K}^*(t)(1+\lambda^2)+\dot{\phi}(t)\mathcal{K}(t)(1-\lambda^2)-2i\lambda\dot{\mathcal{K}}(t)\right]\hat{A}^\dagger(t)\\
&+\frac{i}{2\lambda}\sqrt{\frac{m\omega}{2\lambda\hbar}}\left[\dot{\phi}(t)\mathcal{K}(t)(1+\lambda^2)+\dot{\phi}(t)\mathcal{K}^*(t)(1-\lambda^2)+2i\lambda\dot{\mathcal{K}}^*(t)\right]\hat{A}(t).
\end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we can obtain the diagonalized Hamiltonian thus

$$\begin{aligned}
\frac{\hat{\mathcal{H}}(t)}{\hbar} &= \left(\frac{\beta\Omega(t)}{\delta}\right)\left(\hat{A}^\dagger(t)\hat{A}(t)+\frac{1}{2}\right)\\
&+i\sqrt{\frac{m\omega}{2\lambda\hbar}}\left[\left(\frac{\beta\Omega(t)}{\delta}\right)\left(\mathcal{K}^*(t)\hat{A}(t)-\mathcal{K}(t)\hat{A}^\dagger(t)\right)\right],\\
&+\frac{m\omega\Omega}{8\hbar\lambda^2}\left[\left(\mathcal{K}(t)+\mathcal{K}^*(t)\right)^2+\lambda^2\left(\mathcal{K}(t)-\mathcal{K}^*(t)\right)^2\right]\\
&-\frac{i}{2\lambda\hbar}\bar{E}(t)\left(\mathcal{K}(t)R^*-\mathcal{K}^*(t)R\right).
\end{aligned} \tag{3.5}$$

Here we may point out that if one wishes to solve the equations of motion in the Heisenberg picture. Then the terms free from any dynamical operators in the above equation can be dropped without loss of generality. In absence of the external force $\bar{E}(t)$, (3.5) reduces to

$$\hat{\mathcal{H}}(t)/\hbar = \left(\frac{\beta\Omega(t)}{\delta}\right)\left(\hat{A}^{\dagger}(t)\hat{A}(t)+\frac{1}{2}\right) \tag{3.6}$$

which represents the usual simple harmonic motion with time-dependent frequency $\beta\Omega(t)/\delta$.

4 Constants of the Motion

The use of explicitly time-dependent invariants in applications of quantum theory has received little attention. Presumably, the reason for this lack of attention has been the dearth of examples in which the use of such quantities was both possible and fruitful. However, a class of exact invariants for time-dependent harmonic oscillators, both classical and quantum, was reported [29–31]. The simplicity of the rules for constructing these invariants and the instructive relation of the invariant theory have stimulated an interest in using the invariants for solving some explicit quantum-mechanical problems. In what follows we introduce the constants of motion for the Hamiltonian (2.1) in absence of the driving force [32–35]. Consequently we calculate the eigenvalues and corresponding eigenfunctions of these invariants [36, 37].

4.1 Linear Invariants

We begin by seeking a first-degree invariant [32, 33]

$$\hat{I}(t) = \nu(t)\hat{q} + \mu(t)\hat{p}. \tag{4.1}$$

We require

$$\frac{d\hat{I}}{dt} = \frac{\partial\hat{I}}{\partial t} + \frac{1}{i\hbar}\left[\hat{I},\hat{H}\right] = 0, \tag{4.2}$$

where $\hat{H}$ is the Hamiltonian given by (2.1). From which we see $\nu(t)$ and $\mu(t)$ must satisfy

$$\frac{d\mu}{dt} = \mu\Gamma(t) - \frac{1}{m}\nu, \qquad \frac{d\nu}{dt} = m\omega^2\mu - \nu\Gamma(t). \tag{4.3}$$

In order to find the solution for these equations we eliminate either $\mu(t)$ or $\nu(t)$ to have uncoupled second order differential equation, thus

$$\frac{d^2\nu}{dt^2} + (\omega^2 + \dot{\Gamma}(t) - \Gamma^2(t))\nu = 0, \qquad \frac{d^2\mu}{dt^2} + (\omega^2 - \Gamma^2 - \dot{\Gamma})\mu = 0, \tag{4.4}$$

where over dot indicates to the first derivative. The exact solution for these equations are

$$\nu(t) = \nu(0)f_1(t) + m\omega\mu(0)g_1(t), \qquad \mu(t) = \mu(0)f_2(t) + \frac{1}{m\omega}\nu(0)g_2(t), \tag{4.5}$$

where $f_i(t)$ and $g_i(t), i = 1, 2$ are given by

$$f_1(t) = (\cos\theta(t)\cosh\phi(t) - \lambda\sin\theta(t)\sinh\phi(t)),$$

$$
\begin{aligned}
g_1(t) &= \left(\cos\theta(t)\sinh\phi(t) + \frac{1}{\lambda}\sin\theta(t)\cosh\phi(t)\right),\\
f_2(t) &= \left(\cos\theta(t)\cosh\phi(t) + \frac{1}{\lambda}\sin\theta(t)\sinh\phi(t)\right),\\
g_2(t) &= (\cos\theta(t)\sinh\phi(t) - \lambda\sin\theta(t)\cosh\phi(t)).
\end{aligned}
\tag{4.6}
$$

In this case we are able to construct two class of invariants

$$
\hat{I}^{(p)} = m(\mu\Gamma - \dot{\mu})\hat{q} + \mu\hat{p}, \quad \text{and} \quad \hat{I}^{(q)}(t) = \frac{1}{m\omega^2}(\dot{\nu} + \nu\Gamma(t))\hat{p} + \nu\hat{q}. \tag{4.7}
$$

In the following subsection we turn our attention to construct another different classes of the quadratic invariants.

4.2 Quadratic Invariants

In a similar way we seek a second-degree invariant

$$
\hat{I}(t) = \alpha_1(t)\hat{q}^2 + \beta_1(t)\hat{p}^2 + \gamma_1(t)\hat{q}\hat{p}. \tag{4.8}
$$

From (2.1) and (4.8) together with (4.2) we have

$$
\frac{d\alpha_1}{dt} + 2\alpha_1\Gamma = \gamma_1 m\omega^2, \qquad \frac{d\beta_1}{dt} - 2\Gamma\beta_1 = -\frac{1}{m}\gamma_1, \qquad \frac{d\gamma_1}{dt} = -\frac{2\alpha_1}{m} + 2m\omega^2\beta_1. \tag{4.9}
$$

Now if we use the fact that $\beta_1\alpha_1 = \gamma_1^2/4 + C_0$ where C_0 is a constant and set $\alpha_1 = \sigma^2$, then after simple algebra we obtain the nonlinear differential equation

$$
\ddot{\sigma} + (\omega^2 - \Gamma^2(t) + \dot{\Gamma}(t))\sigma = \frac{m^2\omega^4}{\sigma^3}C_0, \tag{4.10}
$$

which is of the Pinney equation form with solution

$$
\sigma = (ax_1^2 + bx_2^2 + 2cx_1x_2)^{\frac{1}{2}}, \tag{4.11}
$$

where $x_1(t)$ and $x_2(t)$ are linearly independent solutions of the homogeneous equation

$$
\ddot{x} + (\omega^2 + \dot{\Gamma}(t) - \Gamma^2(t))x = 0. \tag{4.12}
$$

The quantity a, b, and c are arbitrary constants subject to the condition $ab - c^2 = (1/w_1)^2$, where w_1 is the Wronskian of the solutions x_1 and x_2 such that

$$
w_1 = x_1\dot{x}_2 - \dot{x}_1x_2. \tag{4.13}
$$

The first class of the quadratic invariant may therefore be expressed in the form

$$
\hat{I}^{(q)}(t) = \left[\sigma\hat{q} + \frac{1}{m\omega_\circ^2}(\dot{\sigma} + \sigma\Gamma(t))\hat{p}\right]^2 + \frac{C_0}{\sigma^2}\hat{p}^2. \tag{4.14}
$$

Similar procedure leads to the second family of invariants. In this case if we write $\beta_1 = \rho^2$ and eliminating α_1 from (4.9), this gives us the Pinney equation

$$
\ddot{\rho} + (\omega^2 - \dot{\Gamma}(t) - \Gamma^2(t))\rho = \frac{C_1}{m^2\rho^3}, \tag{4.15}
$$

and consequently the second class of the invariant can be expressed in the form

$$\hat{I}^{(p)}(t) = \frac{C_1}{\rho^2}\hat{q}^2 + \left[\rho\hat{p} + m(\rho\Gamma(t) - \dot{\rho})\hat{q}\right]^2. \tag{4.16}$$

4.2.1 The Eigenfunctions of the Invariants

We now turn our attention to find the eigenfunctions and the corresponding eigenvalues of the operator $\hat{I}(t)$, however, we start with its eigenstates [36, 37]. The eigenstates of the invariant operator $\hat{I}(t)$ may be found by an operator technique that is completely analogous to the method introduced by Dirac for diagonalizing the Hamiltonian. In this case we define time-dependent canonical lowering and raising operators $\hat{B}$ and $\hat{B}^\dagger$ by the relations

$$\begin{aligned} \hat{B} &= (2\hbar\sqrt{C_0})^{-\frac{1}{2}}\left[\left(\frac{\sqrt{C_0}}{\sigma} - \frac{i}{m\omega^2}(\dot{\sigma} + \sigma\Gamma(t))\right)\hat{p} - i\sigma\hat{q}\right], \\ \hat{B}^\dagger &= (2\hbar\sqrt{C_0})^{-\frac{1}{2}}\left[\left(\frac{\sqrt{C_0}}{\sigma} + \frac{i}{m\omega^2}(\dot{\sigma} + \sigma\Gamma(t))\right)\hat{p} + i\sigma\hat{q}\right]. \end{aligned} \tag{4.17}$$

These operators satisfy the canonical commutation rule $[\hat{B}(t), \hat{B}^\dagger(t)] = 1$. So that the operator $\hat{B}^\dagger\hat{B}$ is a number operator with non-negative integer eigenvalues. The invariant operator given by (4.14) can be written in terms of the operators $\hat{B}(t)$ and $\hat{B}^\dagger(t)$ as

$$\hat{I}^{(q)} = 2\hbar\sqrt{C_0}\left(\hat{B}^\dagger(t)\hat{B}(t) + \frac{1}{2}\right), \tag{4.18}$$

and its eigenstates can be obtained from the coherent state $\exp(\beta\hat{B}^\dagger - \beta^*\hat{B})|0\rangle = |\beta\rangle$ which has the property $\hat{B}(t)|\beta\rangle = \beta(t)|\beta\rangle$, where β is a complex parameter. Similarly, we can define another pair of lowering and raising operators $\hat{D}$ and $\hat{D}^\dagger$ corresponding to the second constant of the motion $\hat{I}^{(p)}$ such that

$$\begin{aligned} \hat{D} &= (2\hbar\sqrt{C_1})^{-\frac{1}{2}}\left[\left(\frac{\sqrt{C_1}}{\rho} + im(\rho\Gamma(t) - \dot{\rho})\right)\hat{q} + i\rho\hat{p}\right], \\ \hat{D}^\dagger &= (2\hbar\sqrt{C_1})^{-\frac{1}{2}}\left[\left(\frac{\sqrt{C_1}}{\rho} - im(\rho\Gamma(t) - \dot{\rho})\right)\hat{q} - i\rho\hat{p}\right]. \end{aligned} \tag{4.19}$$

Since these operators satisfy the relation $[\hat{D}(t), \hat{D}^\dagger(t)] = 1$, therefore the invariant operator in this case becomes

$$\hat{I}^{(p)} = 2\hbar\sqrt{C_1}\left(\hat{D}^\dagger(t)\hat{D}(t) + \frac{1}{2}\right), \tag{4.20}$$

and the eigenstates can also be obtained from the coherent state $\exp(\gamma\hat{D}^\dagger - \gamma^*\hat{D})|0\rangle = |\gamma\rangle$, where γ is complex parameter. To find the eigenfunctions and the corresponding eigenvalue for the constants of the motion $\hat{I}^{(q)}$ and $\hat{I}^{(p)}$ we have to employ the lowering operators $\hat{B}$ and $\hat{D}$. In this case the wave function in the number state corresponding to the operator $\hat{B}$ in terms of the momentum is given by

$$\Psi_s(p,t) = \left(\frac{\sqrt{C_0}}{\pi\hbar\sigma^2}\right)^{\frac{1}{4}}\frac{2^{-s/2}}{\sqrt{s!}}H_s\left(\sqrt{\frac{\sqrt{C_0}}{\hbar\sigma^2}}p\right)$$

$$\times \left(-\frac{1}{2\hbar}\left[\frac{\sqrt{C_0}}{\sigma^2} - \frac{1}{m\omega^2}\left(\Gamma(t) + \frac{d\ln\sigma}{dt}\right)\right]p^2\right), \tag{4.21}$$

where $H_s(.)$ stands for the Hermite polynomial. Alternatively, the wave function corresponding to the coherent state $|\beta\rangle$ can be written thus

$$\Psi_\beta(p,t) = \left(\frac{\sqrt{C_0}}{\pi\hbar\sigma^2}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2}(|\beta|^2 + \beta^2(t))\right)$$
$$\times \exp\left(-\frac{1}{2\hbar}\left[\frac{\sqrt{C_0}}{\sigma^2} - \frac{1}{m\omega^2}\left(\Gamma(t) + \frac{d\ln\sigma}{dt}\right)\right]p^2 + \beta(t)\sqrt{\frac{2\sqrt{C_0}}{\hbar\sigma^2}}p\right). \tag{4.22}$$

For the second constant of motion the wave function in Schrödinger picture can be written in terms of the coordinate as follows

$$\Psi_r(q,t) = \left(\frac{\sqrt{C_1}}{\hbar\pi\rho^2}\right)^{\frac{1}{4}} 2^{-\frac{n}{2}} \frac{1}{\sqrt{r!}} H_r\left(\sqrt{\frac{\sqrt{C_1}}{\hbar\rho^2}}q\right)$$
$$\times \exp\left(-\frac{1}{2\hbar}\left[\frac{\sqrt{C_1}}{\rho^2} + im\left(\tilde{\Gamma} - \frac{d\ln\rho}{dt}\right)\right]q^2\right), \tag{4.23}$$

and the wave function corresponding to the coherent state $|\gamma\rangle$ is of the form

$$\Psi_\gamma(q,t) = \left(\frac{\sqrt{C_1}}{\hbar\pi\rho^2}\right)^{\frac{1}{4}} \exp\left(-\frac{1}{2}\left(|\gamma|^2 + \gamma^2(t)\right)\right)$$
$$\times \exp\left(-\frac{1}{2\hbar}\left[\frac{\sqrt{C_1}}{\rho^2} + im\left(\tilde{\Gamma} - \frac{d\ln\rho}{dt}\right)\right]q^2 + \gamma(t)\sqrt{\frac{2\sqrt{C_1}}{\hbar\rho^2}}q\right). \tag{4.24}$$

It should be noted that the above results can be used to calculate the Green's function and also to consider the remote past as well as the remote future. However, this is not the aim of the present paper. In the next section we discuss the nonclassical properties for the present model from $SU(1,1)$ Casimir operators point of view.

5 Application of $SU(1,1)$ Casimir Operator

It is well known that Lie algebra was used by many researchers to investigate the nonclassical properties of light in quantum-optical systems. They have considered the squeezed states of photons in terms of $SU(1,1)$ and $SU(2)$ Lie algebras and the coherent states associated with these algebras, see for example [38–41]. The squeezed vacuum state is a special case of the Perelomov $SU(1,1)$ coherent state [42, 43], whereas the atomic coherent state (or the Bloch state) is associated with $SU(2)$ Lie algebra [44, 45]. The linear dissipative processes in quantum optical systems can be also studied with $SU(1,1)$ Lie algebra in the framework of the Liouville space formulation [46, 47]. Furthermore, beam splitters [48–50] interferometers [51], and linear directional couplers [52, 53] are successfully described by $SU(2)$ Lie algebra. In these studies the Baker-Campbell-Hausdorff formulas are useful, where in many cases the quantities to be calculated are exponential functions of the generators of the

Lie algebras. Since the main purpose of the present section is to discuss the nonclassical properties of the present system from Lie algebra point of view, we are therefore begin by introducing operators $K_\pm$ and K_z, which satisfy the commutation relations:

$$\left[K_-, K_+\right] = 2\tilde{\sigma} K_z, \qquad \left[K_z, K_\pm\right] = \pm K_\pm, \tag{5.1}$$

where $\tilde{\sigma} = \pm 1$. When $\tilde{\sigma} = 1$, K_+, K_z, and K_- become the generators of the $SU(1,1)$ Lie algebra, and when $\tilde{\sigma} = -1$, K_+, K_z, and K_- become the generators of the $SU(2)$ Lie algebra. The Casimir operator is given by

$$K^2 = K_z^2 - \tilde{\sigma}^{\frac{1}{2}}(K_+K_- + K_-K_+), \tag{5.2}$$

which satisfies

$$\left[K^2, K_\pm\right] = \left[K^2, K_z\right] = 0. \tag{5.3}$$

In what follows we consider the discrete representation of the $SU(1,1)$ Lie algebra and use the state vectors that satisfy

$$\begin{aligned} &K^2|\bar{m};k\rangle = k(k-1)|\bar{m};k\rangle, \qquad K_z|\bar{m};k\rangle = (\bar{m}+k)|\bar{m};k\rangle, \\ &K_+|\bar{m};k\rangle = [(\bar{m}+1)(\bar{m}+2k)]^{\frac{1}{2}}\,|\bar{m}+1;k\rangle, \\ &K_-|\bar{m};k\rangle = [\bar{m}(\bar{m}+2k-1)]^{\frac{1}{2}}\,|\bar{m}-1;k\rangle, \end{aligned} \tag{5.4}$$

where $K_-|0;\bar{m}\rangle = 0$. Here, k is the Bargmann index and $\bar{m}$ is any nonnegative integer. We would like to point out that the Bargmann index k is either $\frac{1}{4}$ or $\frac{3}{4}$ where in $k = \frac{1}{4}$ the basis for the irreducible unitary representation space is a set of states with an even boson number, and for $k = \frac{3}{4}$ the basis is a set of states with an odd boson number.

5.1 Casimir Operator and Nonclassical Properties

We now employ the Casimir operator to describe the Hamiltonian equation (2.1) in absence of the driving force. To achieve our goal we substitute the operator $\hat{a} = (2m\omega\hbar)^{-\frac{1}{2}}(m\omega q + ip)$ with its complex conjugate into the Hamiltonian model, where a and $a^\dagger$ are boson annihilation and creation operators satisfy the commutation relation $[a, a^\dagger] = 1$. In this case we have

$$\frac{\hat{H}}{\hbar} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) - \frac{i\Gamma(t)}{2}\left(\hat{a}^2 - \hat{a}^{\dagger 2}\right). \tag{5.5}$$

Now let us use the $SU(1,1)$ Lie algebra generators to describe the boson annihilation and creation operators. For a single mode bosonic representation the generators $\hat{K}_+$, $\hat{K}_-$, and $\hat{K}_z$ are expressed as

$$\hat{K}_+ = \frac{1}{2}\left(\hat{a}^\dagger\right)^2, \qquad \hat{K}_- = \frac{1}{2}\hat{a}^2, \qquad \hat{K}_z = \frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{5.6}$$

Therefore the Hamiltonian given by (5.5) takes the form

$$\hat{H}/\hbar = 2\omega_\circ\hat{K}_z - i\Gamma(t)(\hat{K}_- - \hat{K}_+). \tag{5.7}$$

Since we are concern with the nonclassical properties, more precisely with the squeezing phenomenon. Therefore it will be more convenient for us to use the generators $\hat{K}_x$, $\hat{K}_y$ and $\hat{K}_z$, where $\hat{K}_x = \frac{1}{2}(\hat{K}_+ + \hat{K}_-)$ and $\hat{K}_y = \frac{1}{2i}(\hat{K}_+ - \hat{K}_-)$ which satisfy the commutation rules

$$\left[\hat{K}_x, \hat{K}_y\right] = -i\hat{K}_z, \qquad \left[\hat{K}_y, \hat{K}_z\right] = i\hat{K}_x, \qquad \left[\hat{K}_z, \hat{K}_x\right] = i\hat{K}_y. \tag{5.8}$$

Then the equations of motion in the Heisenberg picture are

$$\frac{d\hat{K}_x}{dt} = -2\omega\hat{K}_y + 2\Gamma(t)\hat{K}_z, \qquad \frac{d\hat{K}_y}{dt} = 2\omega\hat{K}_x, \qquad \frac{d\hat{K}_z}{dt} = 2\Gamma(t)\hat{K}_x. \tag{5.9}$$

After some manipulations the evolution of the operators$\{\hat{K}_x, \hat{K}_y, \hat{K}_z\}$ can be written thus

$$\begin{pmatrix} \hat{K}_x(t) \\ \hat{K}_y(t) \\ \hat{K}_z(t) \end{pmatrix} = \begin{pmatrix} \mathfrak{f}_1(t) & -\mathfrak{f}_2(t) & -\mathfrak{f}_3(t) \\ \mathfrak{g}_1(t) & \mathfrak{g}_2(t) & \mathfrak{g}_3(t) \\ \mathfrak{h}_1(t) & \mathfrak{h}_2(t) & \mathfrak{h}_3(t) \end{pmatrix} \begin{pmatrix} \hat{K}_x(0) \\ \hat{K}_y(0) \\ \hat{K}_z(0) \end{pmatrix}, \tag{5.10}$$

where

$$\begin{aligned}
\mathfrak{f}_1(t) &= \left(\cos 2\theta(t) - \frac{1}{2\beta^2}\sin^2\theta(t)\right), \\
\mathfrak{f}_2(t) &= \frac{\delta}{\beta}\sin 2\theta(t), \qquad \mathfrak{f}_3(t) = \frac{\delta}{\beta^2}\sin^2\theta(t), \\
\mathfrak{g}_1(t) &= \frac{\delta}{\beta^2}\left(\beta\sin 2\theta(t)\cosh 2\phi(t) + \sin^2\theta(t)\sinh 2\phi(t)\right), \\
\mathfrak{g}_2(t) &= \left(\cos 2\theta(t)\cosh 2\phi(t) + \frac{1}{2\beta}\sin 2\theta(t)\sinh 2\phi(t)\right), \\
\mathfrak{g}_3(t) &= \left(\frac{1}{2\beta}\sin 2\theta(t)\cosh 2\phi(t) + \left[1 + \frac{\sin^2\theta(t)}{2\beta^2}\right]\sinh 2\phi(t)\right), \\
\mathfrak{h}_1(t) &= \frac{\delta}{\beta^2}\left(\beta\sin 2\theta(t)\sinh 2\phi(t) + \sin^2\theta(t)\cosh 2\phi(t)\right), \\
\mathfrak{h}_2(t) &= \left(\cos 2\theta(t)\sinh 2\phi(t) + \frac{1}{2\beta}\sin 2\theta(t)\cosh 2\phi(t)\right), \\
\mathfrak{h}_3(t) &= \left(\frac{1}{2\beta}\sin 2\theta(t)\sinh 2\phi(t) + \left[1 + \frac{\sin^2\theta(t)}{2\beta^2}\right]\cosh 2\phi(t)\right).
\end{aligned} \tag{5.11}$$

The time-dependent arguments $\phi(t)$ and $\theta(t)$ are given by (2.7) and (3.2), respectively.

5.1.1 Squeezing Phenomenon

Having obtained the time-dependent dynamical operators $\hat{K}_x$, $\hat{K}_y$ and $\hat{K}_z$, we are therefore in position to discuss the phenomenon of squeezing. The associated Heisenberg uncertainty relation regarding these operators is given by

$$\langle(\Delta\hat{K}_x)^2\rangle\langle(\Delta\hat{K}_y)^2\rangle \geqslant \frac{1}{4}\left|\langle\hat{K}_z\rangle\right|^2 \tag{5.12}$$

where $\langle(\Delta\hat{K}_j)^2\rangle = \langle\hat{K}_j^2\rangle - \langle\hat{K}_j\rangle^2$.

To measure squeezing, we define the functions

$$F_j = \langle(\Delta\,\hat{K}_j)^2\rangle - \frac{1}{2}\left|\langle\hat{K}_z\rangle\right|, \quad j = x, y. \tag{5.13}$$

Squeezing (reduction) in the fluctuation of $\hat{K}_x$ or $\hat{K}_y$ components occurs if $F_x < 0$ or $F_y < 0$, respectively, and the maximum squeezing is reached when $\langle(\Delta\,\hat{K}_x)^2\rangle = 0$ or $\langle(\Delta\,\hat{K}_y)^2\rangle = 0$. We now proceed by considering one type of $SU(1,1)$ states, namely the Perelomov $SU(1,1)$ coherent state (PCS). The PCS is defined as

$$|\xi_1; k\rangle = (1 - |\xi_1|^2)^k \sum_{\bar{m}=0}^{\infty} \sqrt{\frac{\tilde{\Gamma}(\bar{m}+2k)}{\bar{m}!\tilde{\Gamma}(2k)}}\,\xi_1^{\bar{m}}|\bar{m}; k\rangle, \tag{5.14}$$

where

$$\xi_1 = \tanh\left(\frac{r}{2}\right)\exp(-i\varphi), \quad |\xi_1| \in (0,1),\ r \in (-\infty,\infty),\ \varphi \in (0, 2\pi) \tag{5.15}$$

and $\tilde{\Gamma}$ stands for the gamma function and k is the Bargmann index (where $k(k-1)$ is the eigenvalue of the Casimir operator). For $k = \frac{1}{4}$ and $\frac{3}{4}$, the PCS is the even and the odd parity coherent state, respectively.

5.1.2 Evolution of the Squeezing in the PCS

There is no doubt PCS is special type of squeezed vacuum state [54], which is essentially equivalent to the two-photon coherent state and processes most of the properties of the ordinary coherent states, such as a completeness relation and a reproducing kernel [55]. Also, the PCS can be realized in the framework of the degenerate and nondegenerate parametric amplifier [16–18]. The required quantities for discussing the behavior of $F_x(.)$ or $F_y(.)$ are given by

$$\begin{aligned}
\langle(\Delta K_x(t))^2\rangle &= \frac{k}{2}\left[\left(\mathfrak{f}_3^2(t) + e^{2i\varphi}\mathfrak{f}_+^2(t) + e^{-2i\varphi}\mathfrak{f}_-^2(t)\right)\sinh^2 r\right] \\
&\quad + k\mathfrak{f}_+(t)\mathfrak{f}_-(t)\left(1+\cosh^2 r\right) - \frac{k}{2}\left(e^{i\varphi}\mathfrak{f}_+(t) + e^{-i\varphi}\mathfrak{f}_-(t)\right)\mathfrak{f}_3(t)\sinh 2r, \\
\langle(\Delta K_y(t))^2\rangle &= \frac{k}{2}\left[\left(\mathfrak{g}_3^2(t) + e^{2i\varphi}\mathfrak{g}_-^2(t) + e^{-2i\varphi}\mathfrak{g}_+^2(t)\right)\sinh^2 r\right] \\
&\quad + k\mathfrak{g}_+(t)\mathfrak{g}_-(t)\left(1+\cosh^2 r\right) + \frac{k}{2}\left(e^{i\varphi}\mathfrak{g}_-(t) + e^{-i\varphi}\mathfrak{g}_+(t)\right)\mathfrak{g}_3(t)\sinh 2r, \\
\langle K_z(t)\rangle &= k\left[e^{i\varphi}\mathfrak{h}_-(t) + e^{-i\varphi}\mathfrak{h}_+(t)\right]\sinh r + k\mathfrak{h}_3(t)\cosh r,
\end{aligned} \tag{5.16}$$

where

$$\begin{aligned}
\mathfrak{f}_\pm(t) &= \frac{1}{2}\left[\mathfrak{f}_1(t) \pm i\mathfrak{f}_2(t)\right], \qquad \mathfrak{g}_\pm(t) = \frac{1}{2}\left[\mathfrak{g}_1(t) \pm i\mathfrak{g}_2(t)\right], \\
\mathfrak{h}_\pm(t) &= \frac{1}{2}\left[\mathfrak{h}_1(t) \pm i\mathfrak{h}_2(t)\right].
\end{aligned} \tag{5.17}$$

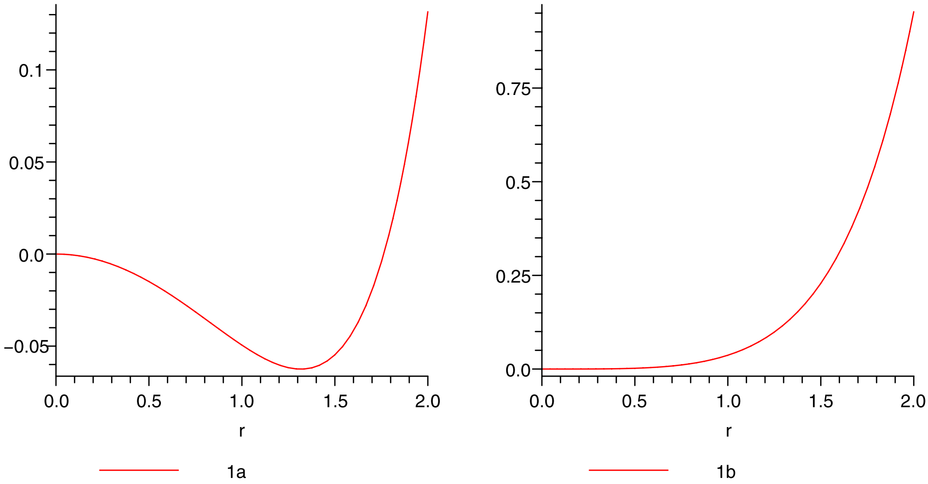

Fig. 1 F_x against r with (**a**) $\varphi = \frac{\pi}{3}$ and (**b**) $\varphi = \frac{\pi}{4}$

From (5.16) it is evident that the quadrature variances $\langle (\Delta K_j(t))^2\rangle$, $j = x, y$ as well as $\langle K_z(t)\rangle$ are always depend on the Bargmann index k. However, this does not affect the phenomenon of squeezing.

At $t = 0$, (5.12), (5.13) and (5.16) lead to the following expressions

$$\begin{aligned} F_x(r,\varphi,t=0) &= 2k\sinh^2\left(\frac{r}{2}\right)\left[\cos^2\varphi\cosh^2\left(\frac{r}{2}\right)-\frac{1}{2}\right],\\ F_y(r,\varphi,t=0) &= 2k\sinh^2\left(\frac{r}{2}\right)\left[\sin^2\varphi\cosh^2\left(\frac{r}{2}\right)-\frac{1}{2}\right]. \end{aligned} \tag{5.18}$$

Therefore the condition for squeezing in the $\hat{K}_x$ or $\hat{K}_y$ component is $F_x < 0$ or $F_y < 0$,

$$1+\cos^2\varphi\sinh^2 r < \cosh r, \quad \text{or} \quad 1+\sin^2\varphi\sinh^2 r < \cosh r, \tag{5.19}$$

respectively.

We now discuss the behavior of the functions F_x and F_y for $t = 0$ at fixed value of the phase φ, such that $\varphi = \pi/3$. In this case we can observe the squeezing is occurred in the quadrature F_x and absence from F_y provided we consider the squeeze parameter r vary between 0 and 2. However, the phenomenon of squeezing disappeared from both quadratures at $\varphi = \pi/4$. This means that the phase φ plays the crucial role to control the squeezing in each quadrature, see Figs. 1a, b and 2a, b. It is also clear from (5.19) that for $\varphi = 0$ there is no squeezing in the quadrature F_x but it occurs in the quadrature F_y and vice versa. For $\varphi = \pi/2$ the squeezing occurs in the quadrature F_x and absence from the second quadrature F_y. We can also examine the case in which $\varphi = \pi/6$ where the phenomenon of squeezing can be seen in the quadrature F_y not in the quadrature F_x. This can be easily deduced from (5.18).

Now we turn our attention to consider the case in which $t > 0$, this means that the fluctuations in the system will take place. For this reason we have plotted some figures to display

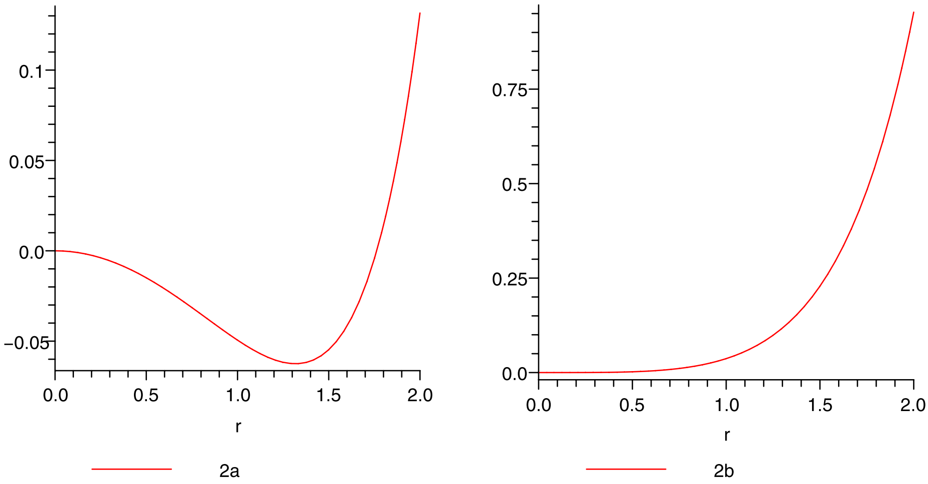

Fig. 2 F_y against r with (**a**) $\varphi = \frac{\pi}{6}$ and (**b**) $\varphi = \frac{\pi}{4}$

this behavior for fixed value of the squeeze parameter $r = 1$ and $\omega = 0.5$. There are two different cases we consider: the first case when we set $\delta = 50$, corresponding to the under critical damping case. The second case when we take $\delta = 0.25$, corresponding to the overcritical damping case. Our study will concentrate on the variation of the phase parameter φ where we discuss the cases in which $\varphi = \pi/4, \pi/3, \pi/2$ and $3\pi/4$. In Fig. 3a we have plotted the quadrature variances F_x (dash line) and F_y (solid line) against the time for $\varphi = \pi/4$ assuming that the system is under critical damping. In this case both quadratures shown regular fluctuations for all period of the time and the phenomenon of squeezing is observed in both quadratures F_x and F_y. Also we realized that the squeezing starts in the first quadrature after onset of the interaction. This is in addition to an exchange between both quadratures where several points of the intersection between them can be seen. Further the maximum value of squeezing is occurred in the first quadrature F_x around ~ -0.35 at the time $t \sim 10.3$. Similar behavior can be reported when we consider the case in which $\varphi = \pi/3$, where the squeezing occurs in the first quadrature F_x faster than that the previous case. However, the second quadrature starts to show squeezing after a short period of the time compare with case $\varphi = \pi/4$, see Fig. 3b. It is interesting to point out that as we increase the value of the phase φ, the patterns of each quadrature is shifted and the maximum value of squeezing occurs faster. This is clearly seen for the case in which $\varphi = \pi/2$ and $3\pi/4$, see Fig. 3c, d. The difference between these two cases is that: for $\varphi = \pi/2$, the phenomenon of squeezing is firstly observed in the quadrature F_x after onset of the interaction. This is followed with a period of an increase in the function corresponding to decreasing in the second quadrature F_y. However, for $\varphi = 3\pi/4$ the squeezing starts in the second quadrature F_y and the observation of an exchange between the two quadratures can also be reported. It should be noted that for all the cases the maximum squeezing always occurs in the first quadrature F_x round the value ~ -0.35, however, there is a time difference between each case. For example when we consider the case in which $\varphi = \pi/3$ the maximum squeezing occurs at the time $t \sim 10$ and for $\varphi = \pi/2$ it occurs at $t = 9.5$, while for $\varphi = 3\pi/4$ the time is $t \sim 8.6$. This means that an increase in the value of the phase angle leads the squeezing faster to reach its maximum. We now turn our attention to consider the overcritical case in which $\delta \varepsilon (-\frac{1}{2}, \frac{1}{2}) - \{0\}$. To do so we have plotted several figures to display this behavior

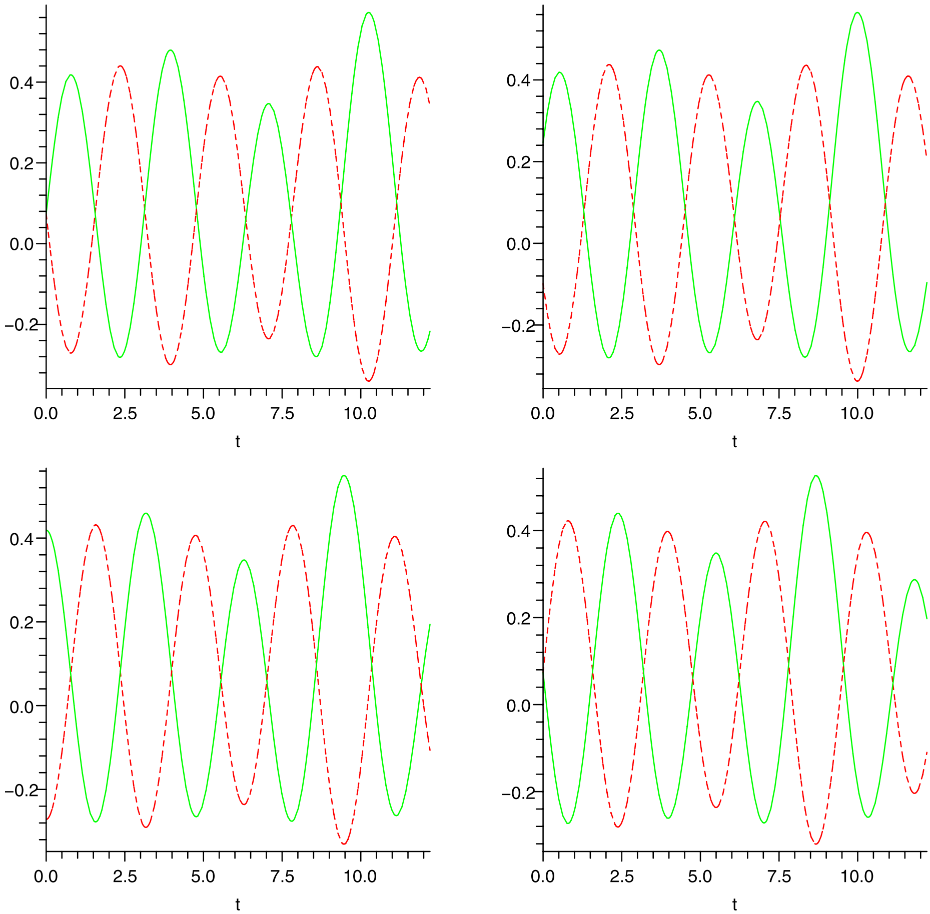

Fig. 3 F_x (*dash line*) and F_y (*solid line*) against the time t with $\delta = 50$, $\omega = 0.5$ $r = 1$ and from *left* to *right* (**a**) $\varphi = \frac{\pi}{4}$, (**b**) $\varphi = \frac{\pi}{3}$, (**c**) $\varphi = \frac{\pi}{2}$ and (**d**) $\varphi = \frac{3\pi}{4}$

for fixed values of $r = 1$, $\delta = 0.25$ and $\omega = 0.5$. For example in Fig. 4a we exhibit the behavior of both quadratures F_x and F_y when $\varphi = \pi/4$. As one can see the phenomenon of squeezing is observed in both quadratures where F_x is slightly decreasing its value to show squeezing after onset of the interaction. However, the function backs again to increase its value and the phenomenon of squeezing disappeared from the rest of the time. On the other hand the quadrature F_y starts to show squeezing after a long period of the time compare with the first quadrature F_x. It has also noted that the maximum squeezing is occurred in the quadrature F_y around -0.75 at the time $t \sim 1.7$. When we consider the case in which $\varphi = \pi/3$ similar behavior to that of $\varphi = \pi/4$ can be reported. However, both quadratures are slightly shifted and the amount of squeezing in the first quadrature F_x decreases while the amount of squeezing in the second quadrature F_y increases, see Fig. 4b. For the case in which $\varphi = \pi/2$ the same behavior can be seen as in the last two cases. For instance more reduction in the amount of squeezing can be observed in the first quadrature F_x compare with the previous case. This is corresponding to increasing in the amount of squeezing in the second quadrature F_y, see Fig. 4c. When we examine the case in which $\varphi = 3\pi/4$, the

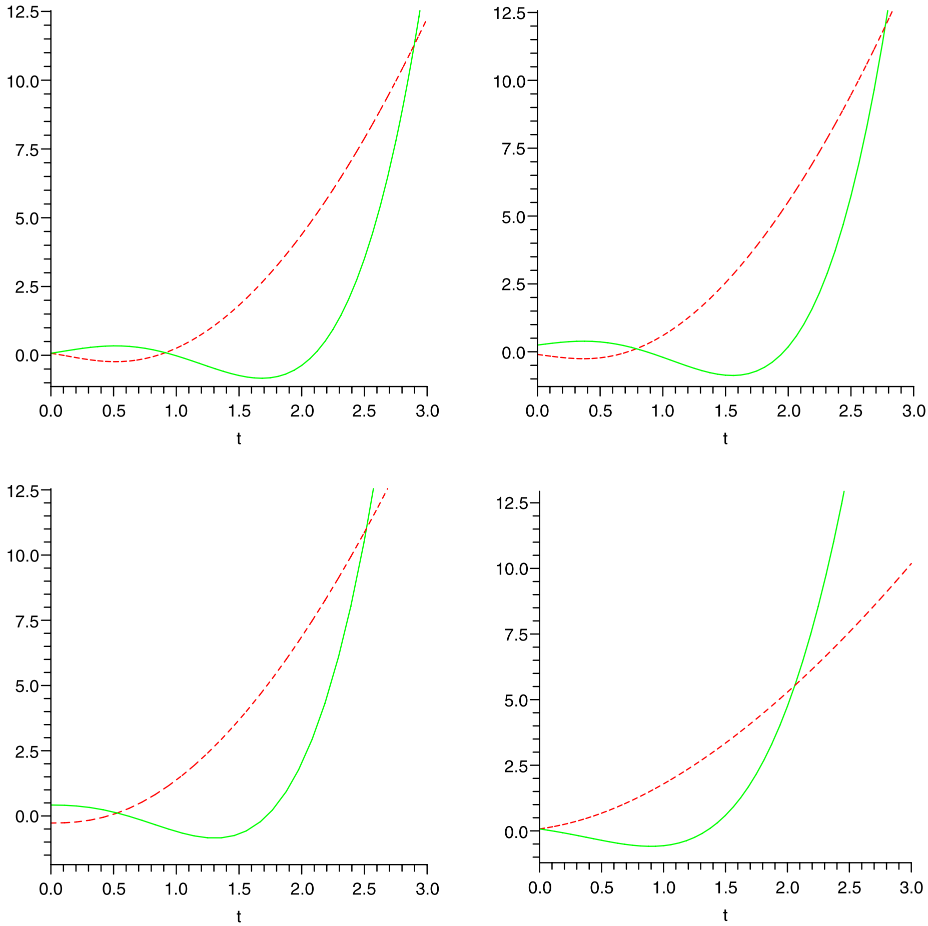

Fig. 4 F_x (*dash line*) and F_y (*solid line*) against the time t with $\delta = 0.25$, $\omega = 0.5$, $r = 1$ and from *left* to *right* (**a**) $\varphi = \frac{\pi}{4}$, (**b**) $\varphi = \frac{\pi}{3}$, (**c**) $\varphi = \frac{\pi}{2}$ and (**d**) $\varphi = \frac{3\pi}{4}$

phenomenon of squeezing is observed in the second quadrature F_y for a long period of the time while the squeezing entirely disappeared from the first quadrature F_x. Also it is noted that the maximum squeezing in this case occurs at the time $t \sim 0.9$ faster than that all the previous cases, see Fig. 4d. Thus from the previous analysis we can conclude that for the under critical damping case the system shown squeezing with periodic exchange between the quadrature variances. However, for the overcritical damping case the squeezing is also observed in both quadratures but without periodicity, which in fact reflects the nature of this case.

Here we emphasis that the *maximum* squeezing is occurred in the first quadrature for the undercritical damping case and in the second quadrature for the overcritical damping case. Finally we would like to point out that although the uncertainty principal (5.12) is always satisfied for all cases, however, the variation in the phase leads to the variation in its behavior. This can be realized from Fig. 5a, b, c, d where we have considered the undercritical damping case and plotted the functions Q_i, $i = 1, 2, 3, 4$ (corresponding to different values

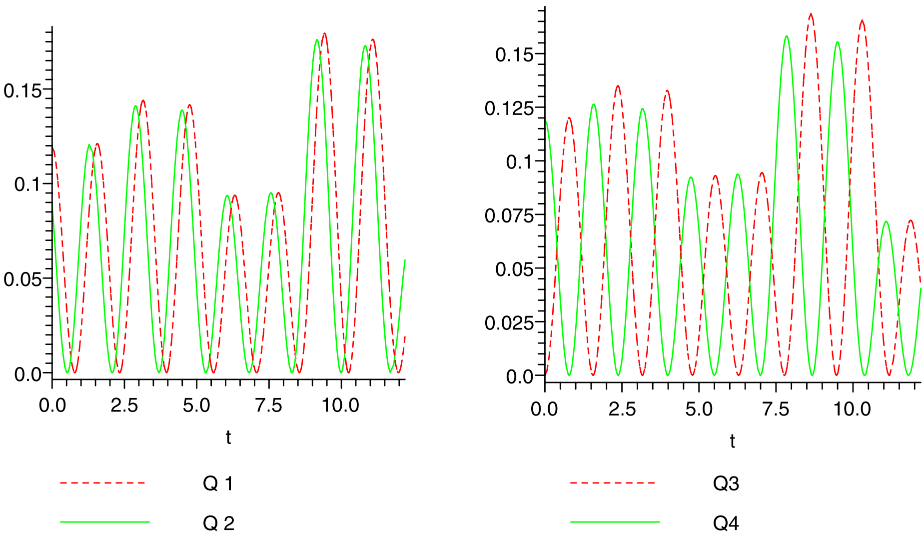

Fig. 5 $Q_i, i = 1, 2, 3, 4$ against the time t with $\delta = 50$, $\omega = 0.5$, $r = 1$, where (**a**) Q_1 corrponds to $\varphi = \frac{\pi}{4}$, (**b**) Q_2 corrponds to $\varphi = \frac{\pi}{3}$, (**c**) Q_3 corrponds to $\varphi = \frac{\pi}{2}$, (**d**) Q_4 corrponds to $\varphi = \frac{3\pi}{4}$

of the phase)

$$Q_i = \langle(\Delta\hat{K}_x)^2\rangle\langle(\Delta\hat{K}_y)^2\rangle - \frac{1}{4}\left|\langle\hat{K}_z\rangle\right|^2, \quad i = 1, 2, 3, 4 \tag{5.20}$$

against the time t, for fixed value of the other parameters. In the first figure, for $\varphi = \pi/4$ and $\pi/3$ we can see a regular fluctuations in the two functions with a slight shifting between them. However, for $\varphi = \pi/2$ and $3\pi/4$ we can also see regular fluctuations but with an exchange between the two functions, see Fig. 5c, d.

6 Conclusion

In the previous sections of the present paper we introduced an alternative model of the damped harmonic oscillator. The system is considered under the influence of an external driving force. We dealt with the problem from quantum mechanics point of view where the wave function for both number and coherent states are obtained. The linear and quadratic invariants are also considered and two classes from each type is introduced. The eigenvalues and the corresponding eigenfunctions are calculated for both number and coherent states representation. Finally we treated the system from $SU(1,1)$ Lie Algebra point view where the Caismir operators are used to discuss the phenomenon of squeezing. It has been shown that the phase angle of Perelomov $SU(1,1)$ coherent state plays the crucial rule of controlling this phenomenon.

Acknowledgement One of us (M.S.A.) is grateful for the financial support from the project Math: 2009/32 of the research centre.

References

1. Caldirola, P.: Nuovo Cimento **18**, 393 (1941)

2. Caldirola, P.: Nuovo Cimento B **77**, 241 (1983)
3. Remaud, B., Hernandez, E.: J. Phys. A, Math. Gen. **13**, 2013 (1980)
4. Colegrave, R.K., Abdalla, M.S.: Opt. Acta **28**, 495 (1981)
5. Colegrave, R.K., Abdalla, M.S.: J. Phys. A, Math. Gen. **14**, 2267 (1981)
6. Colegrave, R.K., Abdalla, M.S.: J. Phys. A **15**, 1549 (1982)
7. Dodonov, V.V., Manko, V.I.: Phys. Rev. A **20**, 550 (1979); and the references therein
8. Leach, P.G.L.: J. Phys. A, Math. Gen. **16**, 3261–3269 (1983)
9. Gzyl, H.: Phys. Rev. A **27**, 2297 (1983)
10. Abdalla, M.S., Colegrave, R.K.: Phys. Rev. A (1985)
11. Abdalla, M.S.: Phys. Rev. A **33**, 2870 (1986)
12. Abdalla, M.S.: Phys. Rev. A **34**, 4598 (1986)
13. Caves, C.M., Schumaker, B.L.: Phys. Rev. A **31**, 3068 (1985)
14. Schumaker, B.L.: Phys. Rep. **135**, 317 (1986)
15. Abdalla, M.S., El-Orany, F.A.A., Perina, J.: IL-Nuovo Cimento B **116**, 137 (2001)
16. Gerry, C.C., Welch, R.R.: J. Opt. Soc. Am. B **8**, 868 (1991)
17. Gerry, C.C.: Phys. Rev. A **37**, 2683 (1988)
18. Gerry, C.C.: J. Opt. Soc. Am. B **8**, 685 (1991)
19. Mostafazadeh, A.: J. Phys. A, Math. Gen. **31**, 6495 (1998)
20. Ban, M.: J. Math. Phys. **33**, 3213 (1992)
21. Ban, M.: Found. Phys. Lett. **5**, 297 (1992)
22. Ban, M.: Opt. Soc. Am. B **10**, 1347 (1993)
23. Abdalla, M.S., Colegrave, R.K.: Lett. Nuovo Cimento **39**, 373 (1984)
24. Abdalla, M.S.: Lett. Nuovo Cimento **44**, 482 (1985)
25. Abdalla, M.S., Al-Gwaiz, M.A.: IL Nuovo Cimento B **105**, 401 (1990)
26. Abdalla, M.S., Colegrave, R.K.: Phys. Rev. A **32**, 1958 (1985)
27. Abdalla, M.S., Colegrave, R.K., Khosravi, A.: IL Nuovo Cimento B **93**, 195 (1986)
28. Abdalla, M.S., Nassar, M.M.: Ann. Phys. **324**, 637 (2009)
29. Lewis, Jr. H.R.: J. Math. Phys. **9**, 1976 (1968)
30. Lewis, Jr. H.R.: Phys. Rev. Lett. **18**, 510 (1967)
31. Lewis, Jr. H.R.: Phys. Rev. Lett. **18**, 636 (1967)
32. Colegrave, R.K., Abdalla, M.S.: J. Phys. A, Math. Gen. **16**, 3805 (1983)
33. Colegrave, R.K., Abdalla, M.S.: J. Phys. A, Math. Gen. **17**, 1567 (1984)
34. Abdalla, M.S., Leach, P.L.G.: J. Phys. A, Math. Gen. **36**, 12205 (2003)
35. Abdalla, M.S., Leach, P.L.G.: J. Phys. A, Math. Gen. **38**, 881–893 (2005)
36. Abdalla, M.S., Leach, P.L.G.: Theor. Math. Phys. **159**, 534 (2009)
37. Abdalla, M.S., Choi, J.-R.: Ann. Phys. **322**, 2795 (2007)
38. Aravind, P.K.: J. Opt. Soc. Am. B **5**, 1545 (1988)
39. Gerry, C.C., Welch, R.R.: J. Opt. Soc. Am. B **9**, 290 (1992)
40. Perelomov, A.M.: Generalized Coherent States and Their Applications. Springer, Berlin (1985)
41. Ban, M.: J. Opt. Soc. Am. B **10**, 1347 (1993); and references therein
42. Perelomov, A.M.: Commun. Math. Phys. **26**, 222 (1972)
43. Perelomov, A.M.: Sov. Phys. Usp. **20**, 703 (1977)
44. Arecchi, F.T., Courtens, E., Gilmore, R., Thomas, H.: Phys. Rev. A **6**, 2211 (1972)
45. Radcliffe, J.M.: J. Phys. A **4**, 313 (1971)
46. Ban, M.: J. Math. Phys. **33**, 3213 (1992)
47. Ban, M.: Found. Phys. Lett. **5**, 297 (1992)
48. Prasad, S., Scully, M.O., Martienssen, W.: Opt. Commun. **62**, 139 (1987)
49. Huttner, B., Ben-Aryeh, Y.: Phys. Rev. A **38**, 204 (1988)
50. Campos, R.A., Saleh, B.E., Teich, M.C.: Phys. Rev. A **40**, 1371 (1989)
51. Yurke, B., McCall, S.L., Klauder, J.R.: Phys. Rev. A **33**, 4033 (1986)
52. Abdalla, M.S., El-Orany, F.A.A., Perina, J.: Acta Phys. Slovaca **50**, 613 (2000)
53. El-Orany, F.A.A., Hassan, S.S., Abdalla, M.S.: J. Opt. B, Quantum Semiclass. **5**, 396 (2003)
54. Wodkiewicz, K., Eberly, J.H.: J. Opt. Soc. Am. B **2**, 458 (1985)
55. Yuen, H.P.: Phys. Rev. A **13**, 2226 (1976)